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FUCHSIAN GROUPS

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

by

Bob Anaya

June 2019

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ABSTRACT

Fuchsian groups are discrete subgroups of isometries of the hyperbolic plane. This thesis will primarily work with the upper half-plane model, though we will provide an example in the disk model. We will define Fuchsian groups and examine their properties geometrically and algebraically. We will also discuss the relationships between fundamental regions, Dirichlet regions and Ford regions. The goal is to see how a Ford region can be constructed with isometric circles.

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Table of Contents

Abstract	iii
Acknowledgements	iv
List of Figures	vi
1 Introduction	1
2 Hyperbolic Geometry	2
2.1 The Hyperbolic Metric	2
2.2 Geodesics	6
2.3 Isometries	8
2.4 Hyperbolic Area and Gauss-Bonnet	11
3 Fuchsian Groups	15
3.1 The Group $\mathrm{PSL}(2, \mathbb{R})$	15
3.2 Discrete and Properly Discontinuous Groups	17
3.3 Algebraic Properties of Fuchsian Groups	20
4 Fundamental Regions	25
4.1 Definition of a Fundamental Region	25
4.2 The Dirichlet Region	28
4.3 Isometric Circles	30
4.4 Computing a Ford Region	34
5 Conclusion	40
Bibliography	41

List of Figures

2.1	The hyperbolic length of a segment	7
2.2	The semicircle L is a geodesic in \mathbb{H}^2	7
2.3	Vertical line L is a geodesic in \mathbb{H}^2	8
2.4	Dilation transformation	9
2.5	Translation transformation	9
2.6	Reflection across the y axis	10
2.7	Inversion across the unit circle	10
2.8	Hyperbolic triangle with 0 ideal points	12
2.9	Hyperbolic triangle with 1 ideal point	12
2.10	Hyperbolic triangle with 2 ideal points	13
2.11	Hyperbolic triangle with 3 ideal points	13
3.1	The axis of the transformation T	17
3.2	Orbit of i for $T(z) = 2z$	19
3.3	Orbit of i for $T(z) = z + 1$	20
3.4	The axis of S and T	24
4.1	Intersection of mutually disjoint regions will be empty	26
4.2	Union of regions for $T \in G$ give \mathbb{H}^2	26
4.3	Union of regions for $T \in G$ will not give \mathbb{H}^2	27
4.4	Regions are not mutually disjoint for $T \in G$	27
4.5	Union of regions for $T \in G$ will give \mathbb{H}^2	28
4.6	Mutually disjoint regions for $T \in G$ will be empty	28
4.7	Hyperbolic half plane containing p	29
4.8	Dirichlet Region centered at $2i$	29
4.9	Dirichlet Region centered at $3i$	30
4.10	Isometric circles for hyperbolic transformation	32
4.11	Isometric circles for elliptic transformation	32
4.12	Isometric circles for orientation preserving transformation of the unit disk	34
4.13	The isometric circles for T and T^{-1}	35
4.14	T maps the circle C to C'	36
4.15	T maps the circle A to A'	37
4.16	T maps the circle B to B'	38

4.17 The Ford Region	39
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Chapter 1

Introduction

Möbius transformations are mappings from the complex plane to itself of the form

$$T(z) = \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{C}$, and $ad - bc \neq 0$. We restrict to $a, b, c, d \in \mathbb{R}$, then T preserves \mathbb{H}^2 . We will use associated matrices of these transformations to help us discover more about the geometric properties hidden in these mappings. We will see the trace of a matrix will determine which transformation is either hyperbolic, elliptic or parabolic. These elements have special properties for Fuchsian groups and their geometries will differ. To begin, we will give some background on hyperbolic geometry, then define what Fuchsian groups are. This will lead us to consider the geometry of fundamental regions, Dirichlet regions and Ford regions. We will see how isometric circles play an important role in the construction of a Ford region.

Chapter 2

Hyperbolic Geometry

This chapter gives preliminary background on hyperbolic geometry. We will see how the hyperbolic length and distance share similarities to Euclidean space. We will also see how isometries and geodesics play an important role in our study. Finally, we will explore hyperbolic area.

2.1 The Hyperbolic Metric

The hyperbolic plane is a less familiar metric space than the Euclidean plane. An introduction to some of the basic properties will be and find it has similarities with the Euclidean plane. There is a one-to-one correspondence between points in \mathbb{R}^2 and the complex plane \mathbb{C} . The notations for the real and imaginary parts of the complex number

$$z = x + iy \in \mathbb{C},$$

to be $\text{Re}(z)=x$ and $\text{Im}(z)=y$. The **conjugate** of z is defined to be

$$\bar{z} = x - iy.$$

The upper half plane model of the **hyperbolic plane** is the metric space consisting of the open half plane

$$\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2; y > 0\} = \{z \in \mathbb{C}; \text{Im}(z) > 0\}.$$

In the upper half plane model, the **hyperbolic length** of a curve, γ , which is parametrized by a differentiable vector valued function

$$t \mapsto (x(t), y(t)), \quad a \leq t \leq b$$

to be

$$h(\gamma) = \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)} dt.$$

This metric is a way of finding the lengths of curves and we illustrate this with an example.

Example 2.1.1. Suppose $P_0 = (x, y_0)$ and $P_1 = (x, y_1)$. To compute the hyperbolic length of the line segment denoted by $[P_0, P_1]$, we will parametrize this segment by, $t \mapsto (x, t)$, $y_0 < t < y_1$. Then,

$$h(\gamma) = \int_{y_0}^{y_1} \frac{\sqrt{0^2 + 1^2}}{t} dt = \frac{1}{t} dt = \ln \frac{y_1}{y_0}.$$

The **hyperbolic distance** between two points z_1 and z_2 is the infimum of the hyperbolic lengths of all piecewise differentiable curves γ going from z_1 to z_2 . It is denoted by

$$d(z_1, z_2) = \inf\{h(\gamma); \gamma \text{ goes from } z_1 \text{ to } z_2\}.$$

Definition 2.1. The set of linear fractional transformations of \mathbb{H}^2 , also known as **Möbius transformations** is of the form

$$\left\{ z \mapsto \frac{az + b}{cz + d} \mid a, b, c, d \in \mathbb{R}, ad - bc = 1. \right\}$$

In addition to preserving circles, angles, and symmetry, these mappings are one-to-one and onto. For each w there is one and only one z that maps to w . This leads us to consider finding the inverse of such mappings. Before we do that, we want to have a simple way to view these transformations. We will use matrices to help give us an algebraic point of view, so we can use it to discover geometric properties.

Definition 2.2. Let $T(z) = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. Then,

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is the matrix associated with T .

Square brackets are used to indicate the matrix is identified with its negative. Since $ad - bc = 1$, every Möbius transformation is invertible. The inverse of a Möbius transformation is the associated inverse matrix of T , which is

$$T^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

These transformations form a group. To convince ourselves, we can show the transformations meet all the requirements to be a group. Composition of Möbius transformations is another Möbius transformation. These transformations have inverses because the determinant is 1 and the identity transformation is simply the associated identity matrix. Finally, the associative property follows since composition of maps is always associative.

Definition 2.3. The *special linear group*, $SL(2, \mathbb{R})$ is the group of 2×2 with determinant 1.

Definition 2.4. The *projective special linear group*, $PSL(2, \mathbb{R}) \cong SL(2, \mathbb{R}) / \{\pm I_2\}$, where I_2 is the identity matrix.

We will now give examples of the associated matrices of Möbius transformations. Each example will have determinant 1.

Example 2.1.2.

$$T(z) = \frac{z+1}{z+2} \iff \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} z$$

Example 2.1.3.

$$T(z) = \frac{\frac{\sqrt{2}}{2}z + \frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}z + \frac{\sqrt{2}}{2}} \iff \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} z$$

Example 2.1.4.

$$T(z) = z + 1 \iff \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} z$$

Example 2.1.5. Let L be a Euclidean circle or a straight line orthogonal to the real axis, which meets the real axis at some finite point α . We would like to show the transformation

$$T(z) = -(z - \alpha)^{-1} + \beta$$

is a Möbius transformation, and for a suitable β maps L to imaginary axis. What this amounts to is showing T is of the form

$$T(z) = \frac{az + b}{cz + d}$$

where $ad - bc = 1$ and $a, b, c, d \in \mathbb{R}$. Rearranging T we find that it is of the form

$$T(z) = \frac{-1}{z - \alpha} + \beta = \frac{\beta z + (-\alpha\beta - 1)}{z - \alpha}$$

and its determinant is 1. Hence, T is a Möbius transformation.

Definition 2.5. Let f be a bijective mapping between two topological spaces. We say if f and its inverse f^{-1} are continuous, then f is said to be a **homeomorphism**.

Theorem 2.6. $PSL(2, \mathbb{R})$ acts on \mathbb{H}^2 by homeomorphisms.

Proof. Let $w = T(z) = \frac{az+b}{cz+d} \in PSL(2, \mathbb{R})$. Then,

$$w = \frac{(az+b)(c\bar{z}+d)}{|cz+d|^2} = \frac{ac|z|^2 + adz + bc\bar{z} + bd}{|cz+d|^2}$$

so that,

$$\text{Im}(w) = \frac{w - \bar{w}}{2i} = \frac{z - \bar{z}}{2i|cz+d|^2} = \frac{\text{Im}(z)}{|cz+d|^2}$$

Since $\text{Im}(z) > 0 \Rightarrow \text{Im}(w) > 0$, this shows $PSL(2, \mathbb{R})$ acts on \mathbb{H}^2 . Since $T(z)$ is continuous and its inverse is continuous, the theorem now follows. \square

Definition 2.7. A transformation of \mathbb{H}^2 onto itself is called an **isometry** if it preserves the hyperbolic distance.

Section 2.3 is dedicated to discuss the isometries of $PSL(2, \mathbb{R})$ much later. However, for the purpose of our next proof, the definition of isometry suffices. The set of all isometries of the upper half plane to be $\text{Isom}(\mathbb{H}^2)$.

Theorem 2.8. $PSL(2, \mathbb{R}) \subset \text{Isom}(\mathbb{H}^2)$

Proof. An application of the chain rule will show this hyperbolic length is independent of the parametrization of γ . Since all transformations in $PSL(2, \mathbb{R})$ map \mathbb{H}^2 to itself, we want to show if $\gamma : I \mapsto \mathbb{H}^2$ is a piecewise differentiable function in \mathbb{H}^2 , then for any matrix

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

in $PSL(2, \mathbb{R})$ we have $h(T(\gamma)) = h(\gamma)$.

$$\frac{dw}{dz} = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} = \frac{1}{(cz+d)^2}$$

$$v = \frac{y}{|cz+d|^2}$$

and hence

$$\frac{dw}{dz} = \frac{v}{y}.$$

Thus,

$$h(T(\gamma)) = \int_0^1 \frac{\left|\frac{dw}{dt}\right| dt}{v(t)} = \int_0^1 \frac{\left|\frac{dw}{dz} \frac{dz}{dt}\right| dt}{v(t)} = \int_0^1 \frac{\left|\frac{dz}{dt}\right| dt}{y(t)} = h(\gamma).$$

□

Definition 2.9. Consider the matrix $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL(2, \mathbb{R})$. Then, $tr(g) = |a + d|$ is defined to be **trace** of g .

A geometrical meaning of the trace function allows us to identify what type of transformation we are working with and immediately know how it acts in the upper half plane.

2.2 Geodesics

In Euclidean geometry, the shortest curve joining two points is the line segment with those two points as endpoints. This subsection defines and describes the shortest curves of the hyperbolic plane, which are also known as geodesics.

Definition 2.10. A **geodesic** between two points in \mathbb{H}^2 is a path of minimal length between them.

Proposition 2.2.1. Two points in \mathbb{H}^2 can be joined by a unique geodesic and the hyperbolic distance between those points is equal to the hyperbolic length of the unique geodesic segment connecting them. This will be denoted by $[z_1, z_2]$, where z_1 and z_2 are in \mathbb{H}^2 .

Theorem 2.11. The geodesics in \mathbb{H}^2 are semicircles and straight lines orthogonal to the real axis.

Proof. (This proof of this theorem can be seen in [Bon09] or [Kat92]).

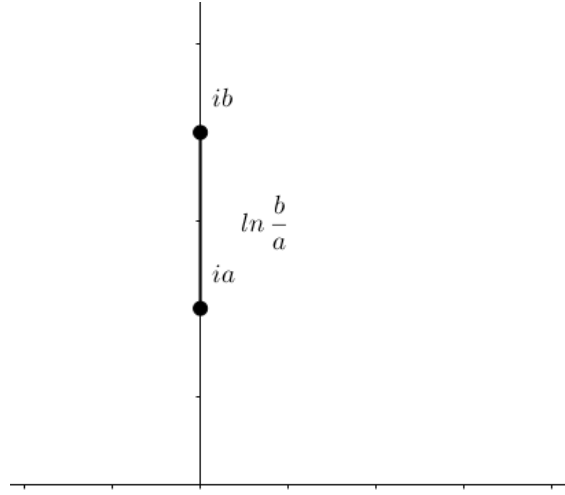


Figure 2.1: The hyperbolic length of a segment

□

Example 2.2.1. Let $L = \{z = e^{i\theta} \mid 0 < \theta < \pi\}$. Then, L is a geodesic orthogonal to the real axis.

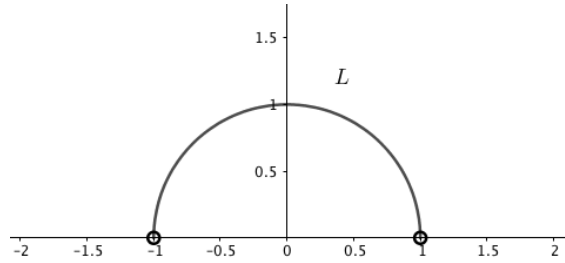


Figure 2.2: The semicircle L is a geodesic in \mathbb{H}^2

Example 2.2.2. Let $L = \{x = 1 \mid y > 0\}$. Then, L is a geodesic and is clearly orthogonal to the real axis.

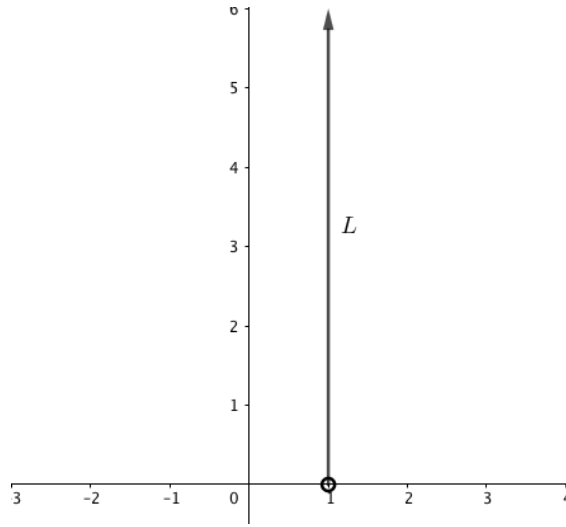


Figure 2.3: Vertical line L is a geodesic in \mathbb{H}^2

Corollary 2.12. *If z_1 and z_2 are two distinct points in \mathbb{H}^2 , then*

$$d(z_1, z_2) = d(z_1, z_3) + d(z_3, z_2)$$

if and only if $z_3 \in [z_1, z_2]$. This is a consequence of the triangle inequality.

Example 2.2.3. Take the example seen in figure 2.1. Suppose there is a point between ia and ib and call it ik , where $a < k < b$. Then, the corollary applies.

2.3 Isometries

The hyperbolic plane has many symmetries and find that it is as symmetric as the Euclidean plane. In this subsection, we will define and describe the isometries of \mathbb{H}^2 . In our discussion, we will explain that all isometries of \mathbb{H}^2 are exactly of the form

$$T(z) = \frac{az + b}{cz + d}$$

or

$$\phi(z) = \frac{-a\bar{z} + b}{-c\bar{z} + d}$$

where $ad - bc = 1$ and $a, b, c, d \in \mathbb{R}$.

Example 2.3.1. Let $\phi : \mathbb{H}^2 \mapsto \mathbb{H}^2$ defined by

$$\phi(x, y) = (kx, ky)$$

This isometry is known as the **homotheties transformation**, which is also known as **dilations**.

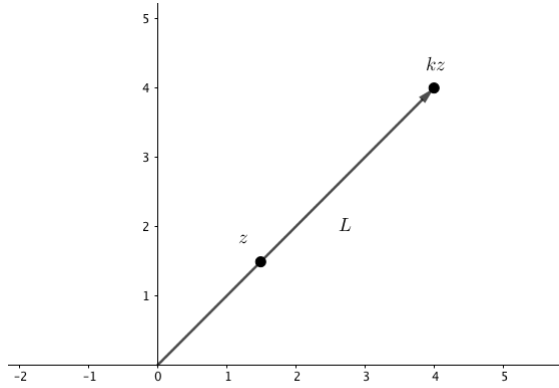


Figure 2.4: Dilation transformation

Example 2.3.2. Let $\phi : \mathbb{H}^2 \mapsto \mathbb{H}^2$ defined by $\phi(x, y) = (x + x_0, y)$, for some $x_0 \in \mathbb{R}$. This is known as the **horizontal translations transformation**, which is also another isometry.

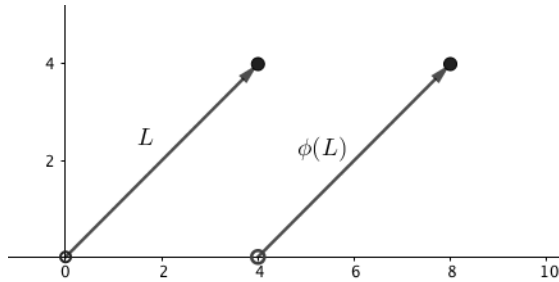
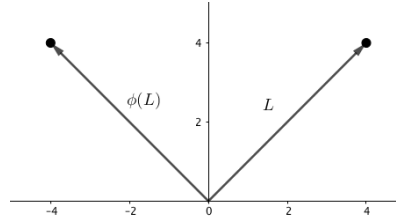


Figure 2.5: Translation transformation

Example 2.3.3. Let $\phi : \mathbb{H}^2 \mapsto \mathbb{H}^2$ defined by $z \mapsto -\bar{z}$. The transformation is **reflection across the y -axis**, which is another isometry of \mathbb{H}^2 .

Figure 2.6: Reflection across the y axis

Example 2.3.4. The **standard inversion**, or simply **inversion**, across the unit circle, which is defined by $\phi(x, y) = (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$ is another isometry. In general, inversion across an arbitrary circle can also be defined as given any point P not the center, the point P' is the inverse to P if

1. P' lies on a ray from K to P , and
2. $KP \cdot KP' = r^2$

where K is the center of the circle and r is the radius of the circle.

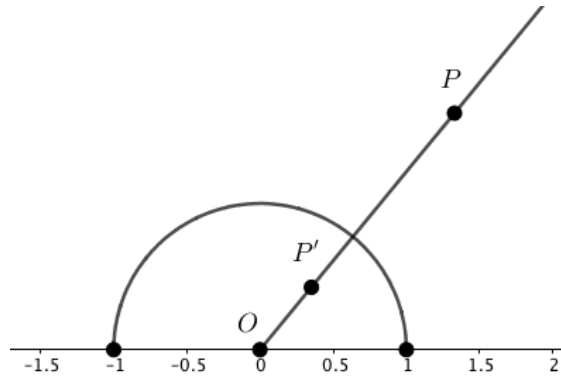


Figure 2.7: Inversion across the unit circle

Definition 2.13. A transformation of \mathbb{H}^2 is called **conformal**, if it preserves angles, and **anti-conformal**, if it preserves the absolute values of angles, but changes the signs.

Example 2.3.5. Homotheties and horizontal translations are conformal transformations because they preserve the angles.

Example 2.3.6. Inversion is an anti-conformal transformation.

2.4 Hyperbolic Area and Gauss-Bonnet

Definition 2.14. For a subset $A \subset \mathbb{H}^2$, we define $\mu(A)$ as the hyperbolic area of A by

$$\mu(A) = \int_A \frac{dx dy}{y^2}$$

if this integral exists.

Theorem 2.15. The hyperbolic area is invariant under all transformations in $PSL(2, \mathbb{R})$: if $A \subset \mathbb{H}^2$, $\mu(A)$ exists, and $T \in PSL(2, \mathbb{R})$, then $\mu(T(A)) = \mu(A)$.

Example 2.4.1. Let's find the hyperbolic area by calculating the integral over the region above the semicircle

$$y = \sqrt{1 - x^2}$$

and between $x = -1$ to $x = 1$. Then,

$$\int_{-1}^1 \int_{\sqrt{1-x^2}}^{\infty} \frac{dx dy}{y^2} = \int_{-1}^1 -\frac{1}{y} \Big|_{\sqrt{1-x^2}}^{\infty} dx.$$

Once we evaluate the integrand, we have

$$\left[\left[\lim_{y \rightarrow \infty} -\frac{1}{y} \right] - \left(-\frac{1}{y} \right) \right] \Big|_{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$$

Now, we evaluate the second integral.

$$\int_{-1}^1 \frac{1}{\sqrt{1-x^2}} dx$$

The function has discontinuities at $x = -1$ and $x = 1$. Consider the two integrals

$$\int_{-1}^0 \frac{1}{\sqrt{1-x^2}} dx$$

and

$$\int_0^1 \frac{1}{\sqrt{1-x^2}} dx$$

We will sum up these integrals to get the area of the region. We take advantage of symmetry and just evaluate one. We see that

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}(x) + C$$

where C is a constant. Therefore,

$$\sin^{-1}(x)|_0^1 = \sin^{-1}(1) - \sin^{-1}(0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$

Then, by symmetry we also have $\frac{\pi}{2}$ for the other integral and therefore the area of the region is π .

A hyperbolic n -sided polygon is a closed set of \mathbb{H}^2 bounded by n hyperbolic geodesic segments. If two line segments intersect, then the point of intersection is called a **vertex** of the polygon. If these vertices are at ∞ or on the real axis, these vertices are known as **ideal points**. If a polygon has vertices only in $\mathbb{R} \cup \{\infty\}$, we say that the polygon is an **ideal polygon**. There are four types of hyperbolic triangles, which depends on how many vertices belong to $\mathbb{R} \cup \{\infty\}$.

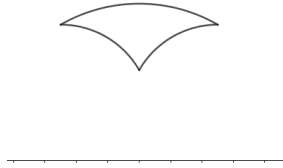


Figure 2.8: Hyperbolic triangle with 0 ideal points

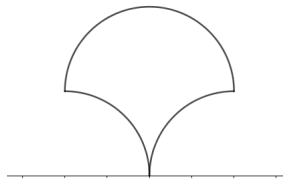


Figure 2.9: Hyperbolic triangle with 1 ideal point

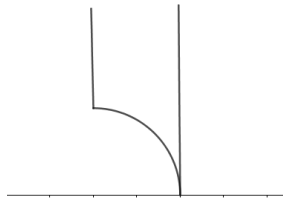


Figure 2.10: Hyperbolic triangle with 2 ideal points

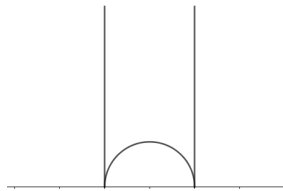


Figure 2.11: Hyperbolic triangle with 3 ideal points

Theorem 2.16. (*Gauss-Bonnet*) Let Δ be the hyperbolic triangle with angles α, β, γ . Then

$$\mu(A) = \pi - \alpha - \beta - \gamma.$$

Before we see some examples, it is important to note that if one of the vertices belongs to $\mathbb{R} \cup \{\infty\}$, then the angle at this vertex will be zero.

Example 2.4.2. Let T be a hyperbolic triangle with vertices at $-1, 0$ and ∞ . The geodesics are the vertical lines -1 and 0 joining to ∞ and the other is the semicircle with 0 joining -1 . Then, all these angles are zero, since all vertices are ideal. This triangle is similar to that of figure 2.11. Then, by the Gauss-Bonnet theorem, we have

$$\mu(T) = \pi - 0 - 0 - 0 = \pi.$$

Since all vertices were in $\mathbb{R} \cup \{\infty\}$, T is an ideal triangle. Example 2.4.1 is an example of an ideal triangle.

Example 2.4.3. Let T be a hyperbolic triangle with 2 vertices in $\mathbb{R} \cup \{\infty\}$. Let us assume one vertex is ∞ and the other is on the real axis. Let the third vertex be at an

angle of $\frac{\pi}{2}$. This triangle is similar to that of figure 2.10. Then, by the Gauss-Bonnet theorem

$$\mu(T) = \pi - \left[\frac{\pi}{2} - 0 - 0 \right] = \frac{2\pi}{2} - \frac{\pi}{2} = \frac{\pi}{2}.$$

Example 2.4.4. Let T be a hyperbolic triangle with 1 vertex in $\mathbb{R} \cup \{\infty\}$. Let us assume the vertex is ∞ . Let the other two vertices have angles of $\frac{\pi}{4}$ and $\frac{\pi}{3}$ respectively. This triangle is similar to that of figure 2.9. Then, by the Gauss-Bonnet theorem

$$\mu(T) = \pi - \left[\frac{\pi}{4} - \frac{\pi}{3} - 0 \right] = \frac{12\pi}{12} - \frac{3\pi}{12} - \frac{4\pi}{12} - 0 = \frac{5\pi}{12}.$$

Chapter 3

Fuchsian Groups

In this chapter, we will define what a Fuchsian group is and what properties these types of groups have. We will also distinguish the elements of $\text{PSL}(2, \mathbb{R})$ by the value of its trace. Furthermore, we will discuss what it means for a Fuchsian group to be discrete and properly discontinuous. Finally, we will discuss algebraic properties of Fuchsian groups.

3.1 The Group $\text{PSL}(2, \mathbb{R})$

There are 3 types of elements in $\text{PSL}(2, \mathbb{R})$ and by the value of its trace we can distinguish which type of transformation it is.

1. If $|Tr(T)| < 2$, then T is an elliptic transformation.
2. If $|Tr(T)| = 2$, then T is a parabolic transformation.
3. If $|Tr(T)| > 2$, then T is a hyperbolic transformation.

Example 3.1.1. Let $T(z) = \frac{z+1}{z+2}$. Based on the value of its trace, T is hyperbolic.

Example 3.1.2. Let $T(z) = z + 1$. Based on the value of its trace, T is parabolic.

Example 3.1.3. Let $T(z) = \frac{\frac{\sqrt{2}}{2}z + \frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}z + \frac{\sqrt{2}}{2}}$. Based on the value of its trace, T is elliptic.

We will now consider finding the fixed points of these transformations. The fixed points are found by solving

$$z = \frac{az + b}{cz + d}$$

with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. The hyperbolic transformation has two fixed points in $\mathbb{R} \cup \{\infty\}$, one *repulsive* and one *attractive*, a parabolic transformation has one fixed point in $\mathbb{R} \cup \{\infty\}$. An elliptic transformation has a pair of conjugate fixed points and therefore, one fixed point in \mathbb{H}^2 . We will illustrate this with a few examples.

Example 3.1.4. Let us find the fixed points of the transformation

$$T(z) = az + b$$

where $a, b \in \mathbb{R}$. So,

$$z = az + b \quad \Rightarrow \quad z - az = b \quad \Rightarrow \quad z(1 - a) = b \quad \Rightarrow \quad z = \frac{b}{1 - a}.$$

Hence, if $a = 1$, then T is parabolic and the only fixed point is ∞ . If $a > 1$, then by the value of the trace, T is hyperbolic and the second fixed point is $\frac{b}{1-a}$.

Example 3.1.5. Let us find the fixed points of the elliptic transformation

$$T(z) = \frac{z \cos \frac{\pi}{2} + \sin \frac{\pi}{2}}{-z \sin \frac{\pi}{2} + \cos \frac{\pi}{2}}.$$

Setting $T(z) = z$ and simplifying T , we find

$$z = \frac{z \cos \frac{\pi}{2} + \sin \frac{\pi}{2}}{-z \sin \frac{\pi}{2} + \cos \frac{\pi}{2}} = \frac{1}{-z} = -\frac{1}{z}.$$

Isolating z and solving for z , we find that

$$z^2 = -1 \quad \Rightarrow \quad z = \pm\sqrt{-1} \quad \Rightarrow \quad z = \pm i.$$

This shows the transformation has a pair of conjugate fixed points and one of them is in \mathbb{H}^2 .

Definition 3.1. A geodesic in \mathbb{H}^2 joining the two fixed points of the hyperbolic transformation T is called the **axis** of T , and we denote it $C(T)$.

Example 3.1.6. Let us find the fixed points of a hyperbolic transformation

$$T(z) = \frac{z + 1}{z + 2}.$$

By setting $T(z) = z$, we have

$$\frac{z + 1}{z + 2} = z \quad \Rightarrow \quad z^2 + 2z = z + 1 \quad \Rightarrow \quad z^2 + z - 1 = 0.$$

By the quadratic formula, we find that the fixed points of T are $z = -\frac{1}{2} + \frac{\sqrt{5}}{2}$ and $-\frac{1}{2} - \frac{\sqrt{5}}{2}$. Therefore, the geodesic connecting these fixed points is the axis of the transformation.

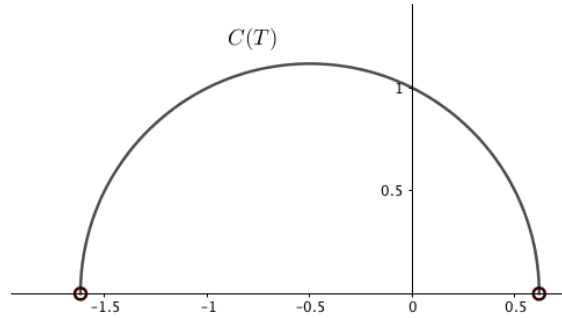


Figure 3.1: The axis of the transformation T

3.2 Discrete and Properly Discontinuous Groups

In this subsection, we define what a Fuchsian group is and describe what it means to be locally finite and properly discontinuous. We also discuss the orbit and stabilizers of Fuchsian groups.

Definition 3.2. A discrete subgroup of $\text{Isom}(\mathbb{H})$ is called a **Fuchsian Group**, if it consists of orientation preserving transformations. A Fuchsian group is a discrete subgroup of $\text{PSL}(2, \mathbb{R})$.

Example 3.2.1. The modular group $\text{PSL}(2, \mathbb{Z})$ is a discrete subgroup of $\text{PSL}(2, \mathbb{R})$ and hence is a Fuchsian group.

Example 3.2.2. The group $\text{PSL}(2, \mathbb{Q})$ is a subgroup of $\text{PSL}(2, \mathbb{R})$, but it is not discrete, therefore is not a Fuchsian group.

Example 3.2.3. The set of integer translations $\{T(z) = z + n \mid n \in \mathbb{N}\}$ is a Fuchsian group.

Example 3.2.4. The set of all translations $\{T(z) = z + b \mid b \in \mathbb{R}\}$ is not a Fuchsian group as it is not discrete.

Definition 3.3. A family $\{M_\alpha \mid \alpha \in A\}$ of subsets of X indexed by elements of a set A is called **locally finite** if for any compact subset $K \subset X$, $M_\alpha \cap K \neq \emptyset$ for only finitely many $\alpha \in A$.

Definition 3.4. For a group G and for $x \in X$, a family $Gx = \{g(x) \mid g \in G\}$ is called the G -**orbit** of the point x . Each point of Gx is contained with a multiplicity equal to the order of G_x , the **stabilizer** of x in G .

Example 3.2.5. Let Γ be the cyclic subgroup of $\text{PSL}(2, \mathbb{R})$ generated by

$$T(z) = \frac{\frac{\sqrt{2}}{2}z + \frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}z + \frac{\sqrt{2}}{2}}$$

Let us find the orbit and stabilizer of i of the elliptic transformation. Since T has order 4, this means the orbit of i can be obtained by evaluating

$$T(i), \quad T^2(i), \quad T^3(i), \quad T^4(i) = i.$$

Since $T(i) = i$, all elements of $\Gamma = \langle T \rangle$ fix i , therefore, the stabilizer is Γ and the orbit of i is i .

Example 3.2.6. Let Γ be the cyclic subgroup of $\text{PSL}(2, \mathbb{R})$ generated by

$$T(z) = 2z.$$

Let us find the orbit and stabilizer of i . Here T has infinite order. Since we also know T is a dilation transformation, we can see i will travel infinitely far in positive y -axis and will travel infinitely close to the origin.

$$\dots T^{-2}(i) = \frac{1}{4}i, \quad T^{-1}(i) = \frac{1}{2}i, \quad T^0(i) = i, \quad T(i) = 2i, \dots$$

Therefore, the stabilizer of i is the identity element, $e \in \Gamma$ and the orbit of i is a subset of the imaginary axis.

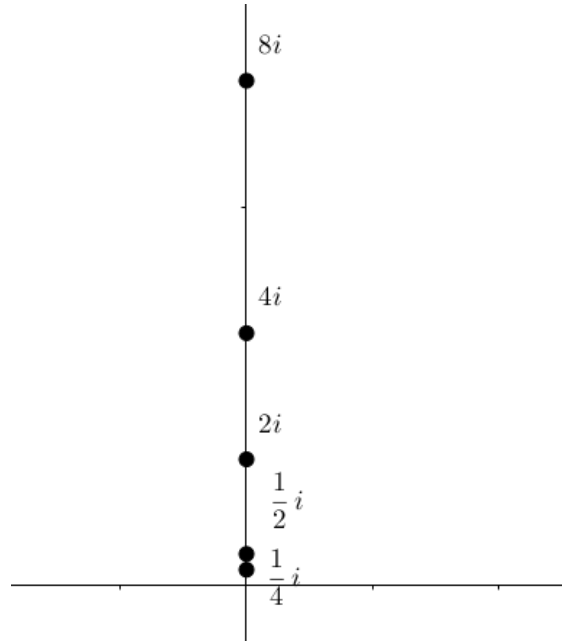


Figure 3.2: Orbit of i for $T(z) = 2z$

Example 3.2.7. Let Γ be the cyclic subgroup of $\text{PSL}(2, \mathbb{R})$ generated by

$$T(z) = z + 1$$

Let us find orbit and stabilizer of i for this transformation. Since T has infinite order, we can get an idea of where T takes i by evaluating

$$\dots T^{-1}(i) = i - 1, \quad T^0(i) = i, \quad T(i) = i + 1, \quad T^2(i) = i + 2, \dots$$

Therefore, the stabilizer of i is the identity element $e \in \Gamma$ and the orbit of i lies on the line $y = 1$.

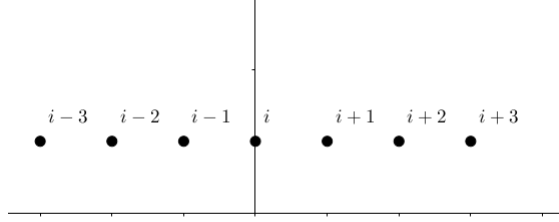


Figure 3.3: Orbit of i for $T(z) = z + 1$

Definition 3.5. We say that a group G acts **properly discontinuously** on X if the G -orbit of any point $x \in X$ is locally finite.

Lemma 3.6. Let G be a group that is properly discontinuous in \mathbb{H}^2 . Then, for each $x \in \mathbb{H}^2$, G_x is finite.

Theorem 3.7. A group G acts properly discontinuously on X if and only if each point $x \in X$ has a neighborhood V such that

$$T(V) \cap V \neq \emptyset$$

for only finitely many $T \in G$.

(The proof of this theorem could be found in [Kat92].)

Theorem 3.8. All hyperbolic and parabolic cyclic subgroups of $PSL(2, \mathbb{R})$ are Fuchsian groups. An elliptic cyclic subgroup of $PSL(2, \mathbb{R})$ is a Fuchsian group if and only if it is finite.

Example 3.2.8. Examples 3.2.3, 3.2.5 and 3.2.6 are Fuchsian groups. To convince ourselves, consider any compact subset \mathbb{H}^2 , call it K , and the intersection any subset of the orbits will be finite. Thus, these cyclic groups act properly discontinuously on \mathbb{H}^2 since the orbits are locally finite.

3.3 Algebraic Properties of Fuchsian Groups

In this subsection, we will take an algebraic point of view to describe Fuchsian groups. We will look at centralizers of parabolic, elliptic and hyperbolic elements of $PSL(2, \mathbb{R})$ and examine their properties.

Definition 3.9. If G is any group and $g \in G$, then the **centralizer** of g in G is defined by

$$C_G(g) = \{h \in G \mid hg = gh\}$$

Lemma 3.10. If $ST = TS$, then S maps the fixed point set of T to itself.

Proof. Suppose that T fixes p , that is, $T(p) = p$. Then

$$S(p) = ST(p) = TS(p)$$

so that $S(p)$ is also fixed by T . □

We now will look at centralizers of parabolic, elliptic and hyperbolic elements of $\text{PSL}(2, \mathbb{R})$.

Example 3.3.1. For a parabolic centralizer, let us consider $T(z) = z + 1$. We would like to find a $S \in \text{PSL}(2, \mathbb{R})$ such that $ST = TS$. By the previous lemma, we know S will map the fixed points of T to itself. Since T is parabolic, this means $S(\infty) = \infty$. Hence, S is of the form

$$S(z) = az + b$$

and $ST = TS$ gives us $a = 1$. Therefore, the S we desire is $S(z) = z + k$, where $k \in \mathbb{R}$.

Example 3.3.2. For a hyperbolic centralizer, let us consider $T(z) = 2z$. Observe $T(0) = 0$ and $T(\infty) = \infty$. We would like to find $S \in \text{PSL}(2, \mathbb{R})$ such that $ST = TS$. Since $S(\infty) = \infty$, S must have the form

$$S(z) = az + b$$

where $a > 1$. Consider

$$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$$

which implies,

$$\begin{bmatrix} 2a & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ 0 & 1 \end{bmatrix}$$

which means this equality is true, if and only if $b = 2b$. Now, this is only possible if $b = 0$. Therefore, the S we desire is $S(z) = (2a)z$, where is $a > 1$.

Example 3.3.3. Let us find an elliptic centralizer for the following transformation. Suppose

$$T(z) = \frac{0z + 1}{-z + 0} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} z$$

This transformation is elliptic since its trace is less than 2. We would like to find a $S \in \text{PSL}(2, \mathbb{R})$ such that $ST = TS$. Consider

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -b & a \\ -d & c \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ -a & -b \end{bmatrix}$$

In order for these matrices to commute, $a = d$ and $-b = c$. Therefore, the centralizer is

$$\begin{bmatrix} a & -c \\ c & -a \end{bmatrix} z$$

or $S(z) = \frac{az-c}{cz-a}$.

Example 3.3.4. Let $T(z) = z + 2$ and $S(z) = z - 1$. We know both transformations are parabolic and fix ∞ . Let us consider their corresponding matrices and show they commute. Consider

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Therefore, by direct calculation, we see these transformations commute.

Example 3.3.5. Let $T(z) = 2z$ and $S(z) = 3z$. We know both transformations are hyperbolic and fix both ∞ and 0 . Let us consider their corresponding matrices and show they commute. Consider

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Since T and S have the same fixed point set, by direct calculation T and S commute.

Example 3.3.6. Let $T(z) = \frac{\frac{\sqrt{3}}{2}z + \frac{1}{2}}{-\frac{1}{2}z + \frac{\sqrt{3}}{2}}$ and $S(z) = \frac{\frac{\sqrt{2}}{2}z + \frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}z + \frac{\sqrt{2}}{2}}$. We know both transformations are elliptic and fix i . Let us consider their corresponding matrices and show they commute.

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4}\sqrt{3}\sqrt{2} & \frac{1}{4}\sqrt{2} \\ \frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{3}\sqrt{2} \end{bmatrix}$$

And the other

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4}\sqrt{3}\sqrt{2} & \frac{1}{4}\sqrt{2} \\ \frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{3}\sqrt{2} \end{bmatrix}$$

Therefore, we have shown the transformations commute, so it must be the case they have the same fixed point set.

Theorem 3.11. *The centralizer in $PSL(2, \mathbb{R})$ of a hyperbolic, parabolic, elliptic element of $PSL(2, \mathbb{R})$ consists of all hyperbolic, parabolic, elliptic elements with the same fixed point set, together with the identity.*

Corollary 3.12. *Two hyperbolic elements in $PSL(2, \mathbb{R})$ commute if and only if they have the same axes.*

Example 3.3.7. Consider again the hyperbolic transformation $T(z) = 2z$. We saw that in example 3.3.2 $S(z) = 3z$ shares the same fix point set as $T(z) = 2z$ and they commute. This means the geodesic connecting the fix points is the vertical line from 0 to ∞ . The vertical line is the axis for both transformations.

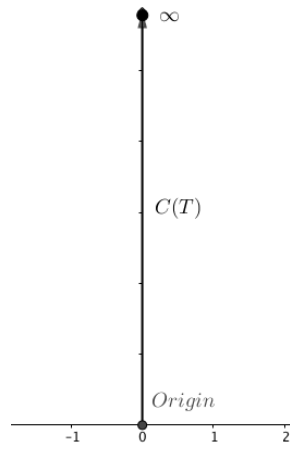


Figure 3.4: The axis of S and T

Chapter 4

Fundamental Regions

In this chapter we examine the properties and geometries of fundamental regions, Dirichlet regions and the Ford region. We will also examine the idea of isometric circles, which can be used to construct these regions.

4.1 Definition of a Fundamental Region

Fundamental regions can be useful to visualize a group structure. These regions can also tell us if an action is discontinuous. The regions also determine the geometry of the quotient space X/Γ . For our purposes, these regions form a tessellation on \mathbb{H}^2 . In other words, these regions can be viewed as a partition of the upper-half plane.

Definition 4.1. A closed region $F \subset X$ is said to be a **fundamental region** for a group G if the following conditions hold:

1. $\bigcup_{T \in G} T(F) = X$
2. $\mathring{F} \cap T(\mathring{F}) = \emptyset$

where F is the closure of a non-empty open set, \mathring{F} called the **interior** of F .

Example 4.1.1. Let $G = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle$, let $F = \{z \in \mathbb{H}^2 \mid 0 \leq \operatorname{Re}(z) \leq 1\}$ and then $\mathring{F} = \{z \in \mathbb{H}^2 \mid 0 < \operatorname{Re}(z) < 1\}$. This is an example of a fundamental region because the union of all images of F is indeed all of \mathbb{H}^2 . The intersection of all interiors \mathring{F} are

empty. Elements of G translate left and right by integer increments as shown in figures 4.1 and 4.2.

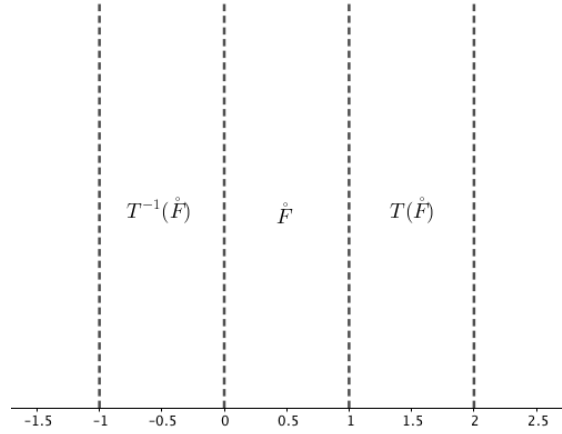


Figure 4.1: Intersection of mutually disjoint regions will be empty

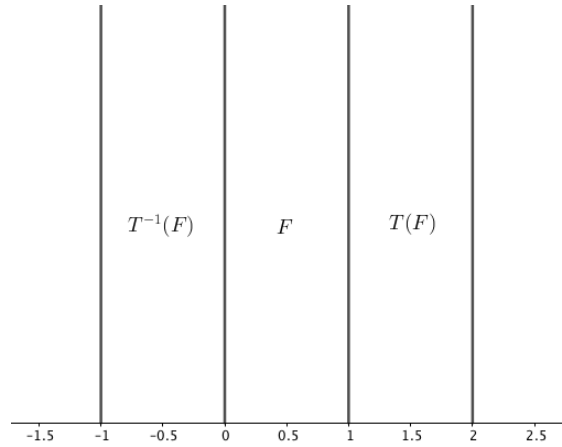


Figure 4.2: Union of regions for $T \in G$ give \mathbb{H}^2

Example 4.1.2. Let $G = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle$, let $F = \{z \in \mathbb{H}^2 \mid 0 \leq \operatorname{Re}(z) \leq \frac{1}{2}\}$ and let $\mathring{F} = \{z \in \mathbb{H}^2 \mid 0 < \operatorname{Re}(z) < \frac{1}{2}\}$. This is not a fundamental region because it fails condition 1. The union of all images with F does not give all of \mathbb{H}^2 .

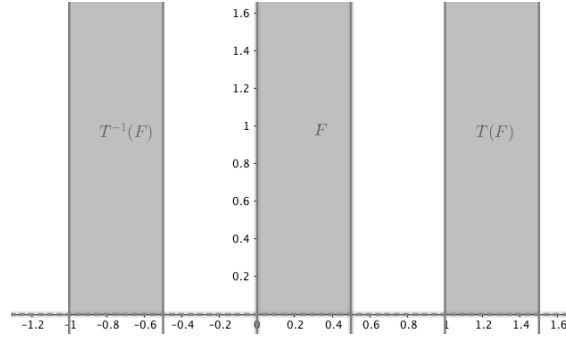


Figure 4.3: Union of regions for $T \in G$ will not give \mathbb{H}^2

Example 4.1.3. Let $G = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle$, let $F = \{z \in \mathbb{H}^2 \mid 0 < \operatorname{Re}(z) < \frac{3}{2}\}$ and let $\mathring{F} = \{z \in \mathbb{H}^2 \mid 0 \leq \operatorname{Re}(z) \leq \frac{3}{2}\}$. This is not a fundamental region because it fails condition 2. The intersection of all interiors with \mathring{F} is not empty. We can see F and $T(F)$ overlap between $1 < \operatorname{Re}(z) < 1.5$.

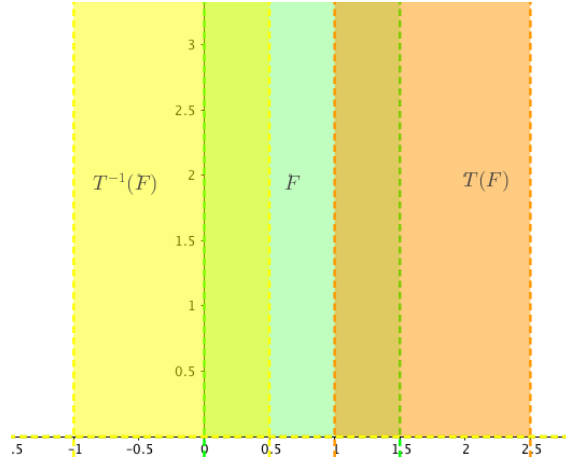


Figure 4.4: Regions are not mutually disjoint for $T \in G$

Example 4.1.4. Let $G = \left\langle \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle$, let $F = \{z = re^{i\theta} \in \mathbb{H}^2 \mid 1 \leq r \leq 2, 0 \leq \theta \leq \pi\}$ and let \mathring{F} be the open set of this region. Then, F is a fundamental region since the union of all the regions for all $T \in G$ will yield \mathbb{H}^2 . The intersection of mutually disjoint regions for $T \in G$ of \mathring{F} will be empty.

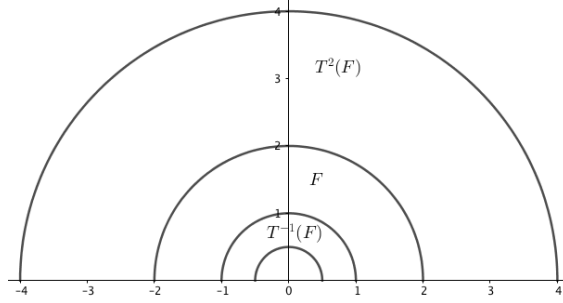


Figure 4.5: Union of regions for $T \in G$ will give \mathbb{H}^2

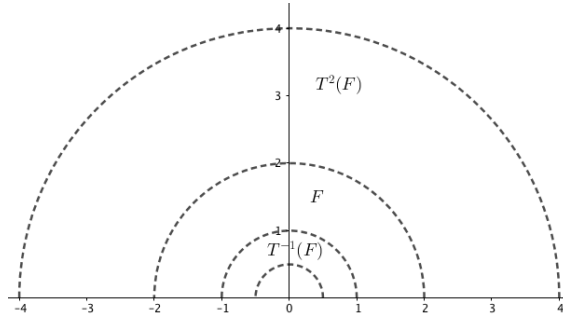


Figure 4.6: Mutually disjoint regions for $T \in G$ will be empty

4.2 The Dirichlet Region

Let Γ be an arbitrary Fuchsian Group and let $p \in \mathbb{H}^2$ be not fixed by any element of $\Gamma - \{Id\}$. We will define the **Dirichlet region** for Γ centered at p to be the set

$$D_p(\Gamma) = \{z \in \mathbb{H}^2 \mid d(z, p) \leq d(z, T(p)) \text{ for all } T \in \Gamma\}$$

Definition 4.2. A *perpendicular bisector* of the geodesic segment $[z_1, z_2]$ is the unique geodesic through w , the midpoint of $[z_1, z_2]$, orthogonal to $[z_1, z_2]$.

Definition 4.3. We will denote the perpendicular bisector of the geodesic segment $[p, T(p)]$ by $L_p(T)$ and the hyperbolic half plane containing p is denoted by $H_p(T)$. Therefore, an equivalent definition for the Dirichlet region is given by

$$D_p(\Gamma) = \cap H_p(T)$$

for $T \in \Gamma$ and $T \neq Id$.

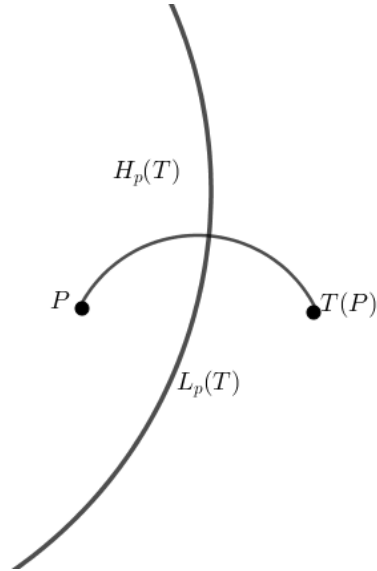


Figure 4.7: Hyperbolic half plane containing p

Example 4.2.1. Let $\Gamma = \text{PSL}(2, \mathbb{Z})$ and let $p = 2i$. Observe p is not fixed by any element other than the identity in Γ . Let $F = \{z \in \mathbb{H}^2 \mid |z| \geq 1, |Re(z)| \leq \frac{1}{2}\}$. Then, F is a Dirichlet region.

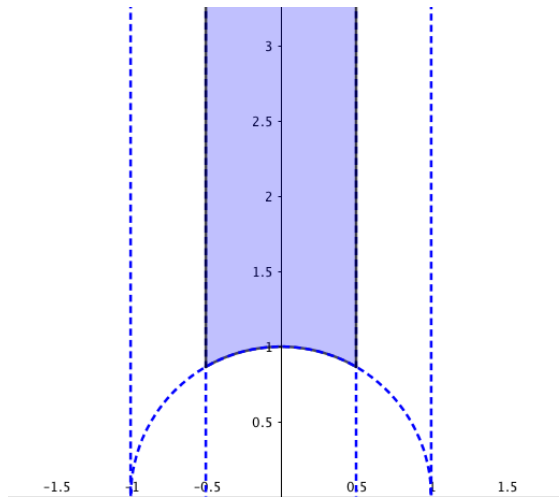


Figure 4.8: Dirichlet Region centered at $2i$

We note that F has three geodesic “sides” of F are $L_p(T)$, $L_p(T^{-1})$ and $L_p(S)$, where T and S are the isometries.

$$T(z) = z + 1, \quad S(z) = -\frac{1}{z}$$

To see this, we apply the transformations T and its inverse T^{-1} to the point $2i$. T maps $2i$ to $2i + 1$ and $2i - 1$ respectively. The next step is to take the perpendicular bisector of the segments $[2i, 2i - 1]$ and $[2i, 2i + 1]$. The perpendicular bisectors cut these segments at $\operatorname{Re}(z) = -\frac{1}{2}$ and $\operatorname{Re}(z) = \frac{1}{2}$. Finally, we want to shade the half plane containing $2i$ with respect to the bisectors. If we continue to apply T to $2i$, we want to take the intersection of the shaded regions for all T . Similarly, we do the same with the transformation S . Observe that S is of order 2, which means S^2 gives the corresponding identity transformation. Here, S takes $2i$ to $\frac{i}{2}$. Thus, we want the perpendicular bisector of the segment $[\frac{i}{2}, 2i]$. The midpoint of the segment is i and we want to shade the region above the circular arc containing i .

Example 4.2.2. Let $\Gamma = \langle T \mid T(z) = 2z \rangle$ and let $p = 3i$. Then, the Dirichlet region centered at p is

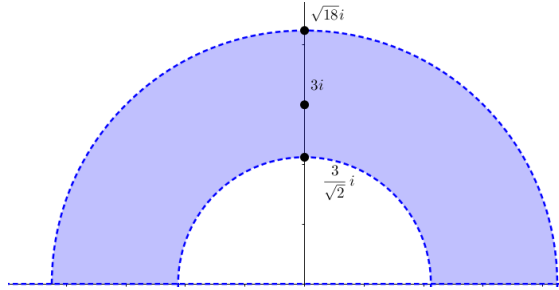


Figure 4.9: Dirichlet Region centered at $3i$

4.3 Isometric Circles

In this subsection, we define isometric circles and examine their properties. We also will see geometrically how isometric circles of a transformation T act with the isometric circles of T^{-1} . We will then transition to examining the isometric circles of the unit disk because the model provides a convenient way to compute the Ford Region.

Let $T(z) = \frac{az+b}{cz+d} \in \text{PSL}(2, \mathbb{R})$. Since $T'(z) = (cz+d)^{-2}$, the locally Euclidean lengths are scaled by $|T'(z)| = |cz+d|^{-2}$. Thus, locally Euclidean area is scaled by $|cz+d|^{-4}$. The Euclidean areas of regions are not altered in magnitude if and only if $|cz+d| = 1$.

Proposition 4.3.1. Let $T(z) = \frac{az+b}{cz+d}$ with $c \neq 0$, then locus of such a z is a circle

$$I(T) = \{z \in \mathbb{C} \mid |cz+d| = 1\}$$

where the center is $-\frac{d}{c}$ and radius $\frac{1}{|c|}$.

Proof. We will show $I(T)$ is a circle with center $-\frac{d}{c}$ and radius $\frac{1}{|c|}$.

Consider $|cz+d| = 1$. Then,

$$\begin{aligned} \Rightarrow |z + \frac{d}{c}| &= \frac{1}{|c|} \\ \Rightarrow |(x+iy) + \frac{d}{c}| &= \frac{1}{|c|} \\ \Rightarrow |(x + \frac{d}{c}) + iy| &= \frac{1}{|c|} \\ \Rightarrow \sqrt{(x + \frac{d}{c})^2 + y^2} &= \sqrt{\frac{1}{c^2}} \\ \Rightarrow (x + \frac{d}{c})^2 + y^2 &= \frac{1}{c^2} \end{aligned}$$

This takes the form of a Euclidean circle with center $(-\frac{d}{c}, 0)$ and radius $\frac{1}{|c|}$. \square

Corollary 4.4. Let $T^{-1}(z) = \frac{-dz+b}{cz-a}$ with $c \neq 0$, then locus of such a z is a circle

$$I(T^{-1}) = \{z \in \mathbb{C} \mid |cz-a| = 1\}$$

where the center is $\frac{a}{c}$ and radius $\frac{1}{|c|}$.

Definition 4.5. These $I(T)$ are called **isometric circles**.

We point out the radius of the isometric circles of a transformations T and T^{-1} are equal. Let us consider some examples and see how hyperbolic and elliptic elements differ with their respective isometric circles.

Example 4.3.1. Let $T(z) = \frac{z+1}{z+2}$ be the hyperbolic transformation. Then,

$$I(T) = \{z \in \mathbb{C} \mid |z+2| = 1\}$$

where the center is -2 and the radius is 1.

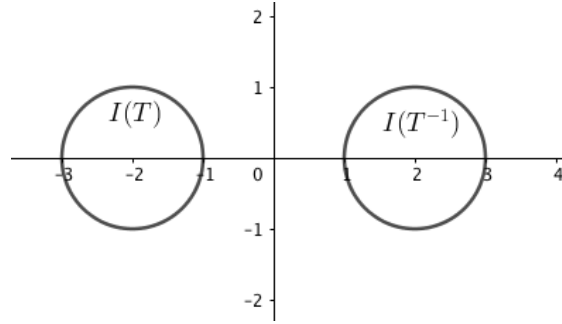


Figure 4.10: Isometric circles for hyperbolic transformation

Example 4.3.2. Let $T(z) = \frac{\frac{\sqrt{2}}{2}z + \frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}z + \frac{\sqrt{2}}{2}}$ be the elliptic transformation. Then,

$$I(T) = \left\{ z \in \mathbb{C} \mid |z - 1| = \frac{1}{|-\frac{\sqrt{2}}{2}|} = \sqrt{2} \right\}$$

where the center is 1 and the radius is $\sqrt{2}$.

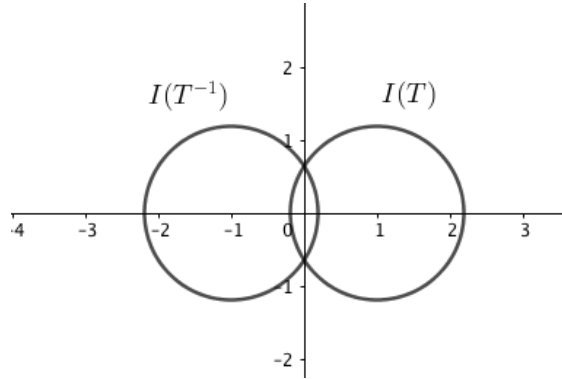


Figure 4.11: Isometric circles for elliptic transformation

Example 4.3.3. Let $T(z) = z + 1$. Observe

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} z.$$

We see that T is parabolic with $c = 0$. This means there is no unique circle with the isometric property since ∞ is a fixed point.

In these examples, we see geometrically how isometric circles of T act with the isometric circles of T^{-1}

1. When $I(T)$ and $I(T^{-1})$ intersect, T is elliptic.
2. When $I(T)$ and $I(T^{-1})$ do not intersect, T is hyperbolic.

Definition 4.6. *The **unit disk** is defined to be*

$$\mathbb{B}^2 = \{z \in \mathbb{C} \mid |z| < 1\}.$$

The map

$$f(z) = \frac{zi + 1}{z + i}$$

is a 1-1 map of \mathbb{H}^2 and provides an isometry onto \mathbb{B}^2 .

We now turn our attention away from the upper half-plane model to the unit disk model. The properties of isometric circles are preserved in the disk model. The motivation behind the switch is to be able to use the isometric circles in the disc model to construct Ford Regions. The disk model provides a simple way to compute these regions. First, we will see an example and observe how these isometric circles act in the disk model.

Proposition 4.3.2. The group of orientation preserving isometries of the unit disk given by the matrices

$$T(z) = \frac{az + \bar{c}}{cz + \bar{a}}$$

where $a, c \in \mathbb{C}$ and $a\bar{a} - c\bar{c} = 1$.

Example 4.3.4. Let $T(z) = \frac{3z+(2-2i)}{(2+2i)z+3}$ be an orientation preserving transformation of the unit disk. Then,

$$I(T) = \{z \in \mathbb{C} \mid |(2+2i)z+3| = 1\}$$

where the center is

$$-\frac{3}{2+2i} = -\frac{6-6i}{8} = -\frac{3}{4} + \frac{3}{4}i$$

and the radius is

$$\frac{1}{|2+2i|} = \frac{1}{\sqrt{8}}$$

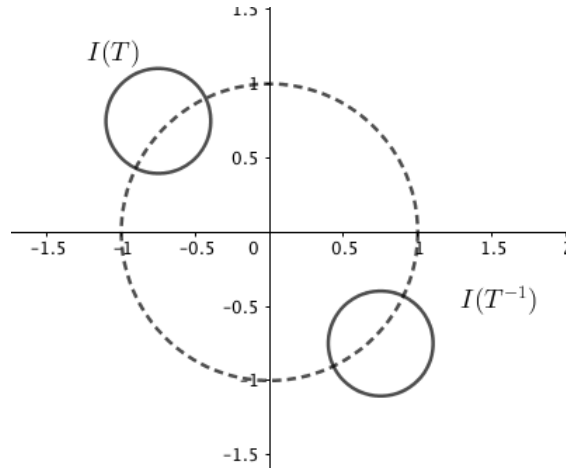


Figure 4.12: Isometric circles for orientation preserving transformation of the unit disk

The rest of this section Γ will be a discrete group of orientation preserving isometries of the unit disk. We will also assume $c \neq 0$ for all elements in the group Γ .

Definition 4.7. *The closure of the set of points in the unit disc, which are exterior to the isometric circles of all transformations in the group Γ to be the **Ford fundamental region**.*

4.4 Computing a Ford Region

In this subsection, we compute a Ford Region with an explicit example. A Ford Region is a variation of the Dirichlet Region, which is a Fundamental region. We translate Dirichlet Regions to the Disc model. The purpose for this transition is the convenient it provides for performing computations and visualizing distances.

Theorem 4.8. *Any orientation preserving isometry T of the unit disk is an inversion in $I(T)$ followed by a reflection in the straight line L , the Euclidean bisector between the centers of the isometric circles $I(T)$ and $I(T^{-1})$.*

This theorem is useful since inversion is just a reflection. Inversion in the unit disk followed by another reflection is a rotation. The strategy is to use rotational matrices to produce isometric circles. The arcs of these isometrics circles become the sides of the hyperbolic octagon we will construct. The group will be generated by 4 elements.

Example 4.4.1. Consider the corresponding matrix of the orientation preserving transformation

$$T = \begin{bmatrix} -1 + i - \left(\frac{1}{2} - \frac{i}{2}\right) \sqrt{2} & \frac{1-i}{\sqrt{\sqrt{2}-1}} \\ \frac{1+i}{\sqrt{\sqrt{2}-1}} & -1 - i - \left(\frac{1}{2} + \frac{i}{2}\right) \sqrt{2} \end{bmatrix}$$

The corresponding isometric circle has center $\sqrt{\frac{1+\sqrt{2}}{2}}$ and the radius is $\sqrt{\frac{1}{2\sqrt{2}+2}}$.

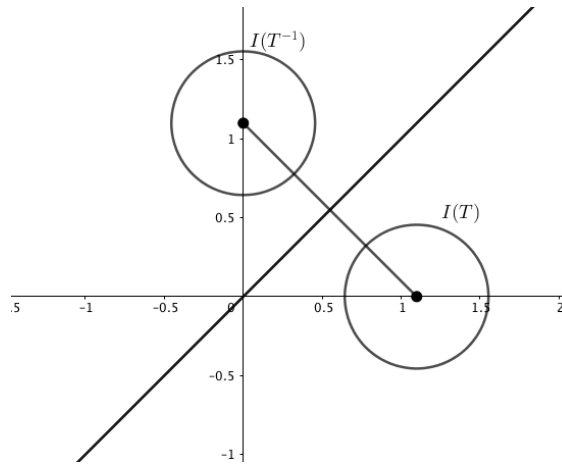


Figure 4.13: The isometric circles for T and T^{-1}

The transformation that takes the isometric circle $I(T)$ to $I(T^{-1})$ is the one in example 4.4.1 by theorem 4.7. Let C be $I(T)$ rotated by $\frac{\pi}{4}$ and let C' be $I(T^{-1})$ rotated by $\frac{\pi}{4}$. We want to find a transformation that maps C to C' . We care for these circles because they help form a side-pairing to construct the region. To find the transformation that maps C to C' , we will consider the rotational matrices

$$M(z) = \begin{bmatrix} e^{-\frac{i\pi}{8}} & 0 \\ 0 & e^{\frac{i\pi}{8}} \end{bmatrix} z = e^{-\frac{i\pi}{4}} z$$

and its inverse

$$M^{-1}(z) = \begin{bmatrix} e^{\frac{i\pi}{8}} & 0 \\ 0 & e^{-\frac{i\pi}{8}} \end{bmatrix} z = e^{\frac{i\pi}{4}} z$$

Since rotation is an isometry, we will consider the matrix, $M^{-1}TM$. This transformation takes the circle C to the circle C' . To see this, we apply M to the circle C , which just rotates it back to $I(T)$ by $-\frac{\pi}{4}$. Applying T maps C to $I(T^{-1})$. Finally, applying the transformation M^{-1} to C rotates it by $\frac{\pi}{4}$ and maps it to C' as seen in figure 4.14. We note the Euclidean bisector between the centers of C and C' is the y -axis.

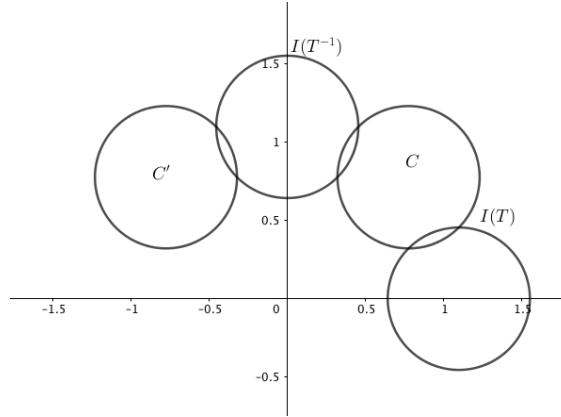


Figure 4.14: T maps the circle C to C'

Continuing with this idea, let us consider

$$N(z) = \begin{bmatrix} e^{-\frac{i\pi}{2}} & 0 \\ 0 & e^{\frac{i\pi}{2}} \end{bmatrix} z = e^{-i\pi} z$$

and its inverse

$$N^{-1}(z) = \begin{bmatrix} e^{\frac{i\pi}{2}} & 0 \\ 0 & e^{-\frac{i\pi}{2}} \end{bmatrix} z = e^{i\pi} z.$$

The transformation $N^{-1}TN$ takes circle A to circle A' . To see this, we apply N to the circle A . This maps A back to $I(T)$ by $-\pi$. Then, applying T maps A to $I(T^{-1})$. Finally, applying N^{-1} maps A to A' by rotation of π . These circles form another side-pairing. The Euclidean bisector of the centers of A and A' is the line $y = x$.

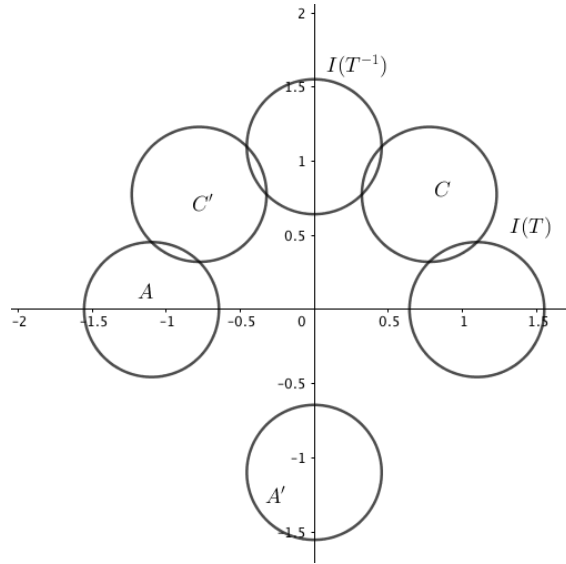


Figure 4.15: T maps the circle A to A'

Finally, consider rotation transformation

$$F(z) = \begin{bmatrix} e^{-\frac{i5\pi}{8}} & 0 \\ 0 & e^{\frac{i5\pi}{8}} \end{bmatrix} z = e^{-\frac{5\pi i}{4}} z$$

and its inverse

$$F^{-1}(z) = \begin{bmatrix} e^{\frac{i5\pi}{8}} & 0 \\ 0 & e^{-\frac{5i\pi}{8}} \end{bmatrix} z = e^{\frac{5\pi i}{4}} z.$$

Now, the following transformation $F^{-1}TF$ maps B to B' . By the same reasoning, we apply F to the circle B and this maps B to $I(T)$ by rotation of $-\frac{5\pi}{4}$. Then, we apply T , this maps B to $I(T^{-1})$. Finally, by applying F^{-1} , this maps B to B' by rotation of $\frac{5\pi}{4}$. This forms the last side-pairing and the Euclidean bisector of the centers of B and B' is the y -axis.

Since there is an isometry mapping \mathbb{H}^2 to the unit disk, the hyperbolic octagon in the unit disk have angles of $\frac{\pi}{4}$. This is because the angle at the origin is $\frac{\pi}{4}$ and by the Gauss-Bonnet theorem, the other two angles are $\frac{\pi}{8}$. Furthermore, the area of each of the hyperbolic triangles is $\frac{\pi}{2}$.

Theorem 4.10. *Let $\{T_i\}$ be a subset of Γ consisting of those elements which pair the sides of some fixed Dirichlet Region F . Then, $\{T_i\}$ is a set of generators for Γ .*

What we have in figure 4.17 is a Dirichlet Region in the unit disk by theorem 4.9. Thus, the side-pairing transformations generate the entire group. The transformations T , $M^{-1}TM$, $N^{-1}TN$ and $F^{-1}TF$ are the corresponding generators by theorem 4.10. Therefore, by definition, the Ford region is the closure set of points that lie outside the isometric circles inside the unit disk of all transformations. The Ford region is shown below.

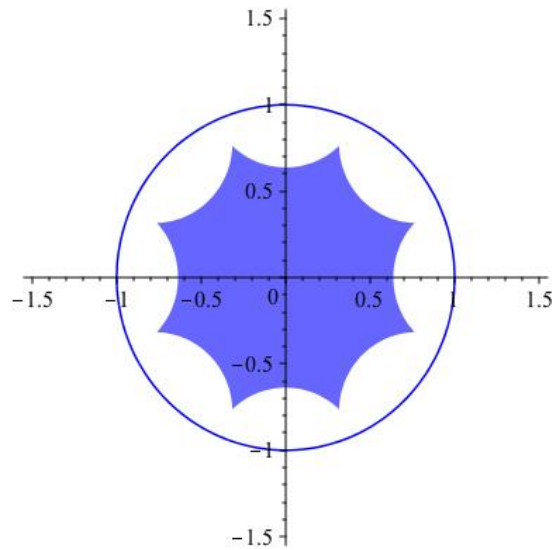


Figure 4.17: The Ford Region

We see that this figure is a hyperbolic octagon. Since all eight sides are arcs of the circles of the same Euclidean radius of equal Euclidean length, the sides identified by a generator are the isometric circles of this generator and its inverse. This hyperbolic octagon gives hyperbolic structure on genus 2 surface.

Chapter 5

Conclusion

In this thesis we began discussing the hyperbolic metric. We calculated hyperbolic lengths and distances. We also discussed Möbius transformations and how these transformations can be looked at algebraically by their associated matrices. The trace of a matrix determined which element in $\mathrm{PSL}(2, \mathbb{R})$ are hyperbolic, elliptic or parabolic.

We discussed how to find the fixed points of these elements. We defined what Fuchsian groups were and gave examples of them geometrically and algebraically. We discussed what it meant to be properly discontinuous and showed examples of cyclic generated groups and their orbits and stabilizers. We saw some algebraically properties of Fuchsian groups and it was shown that elements of $\mathrm{PSL}(2, \mathbb{R})$ commute if and only if they share the same fixed point set.

We finally looked at some examples of fundamental regions and Dirichlet regions. We then defined what a Ford domain was and gave an example of one. We saw how the relationship between isometric circles and the orientation preserving transformations helped in the construction of the Ford domain.

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